

Derivatives of Schur, Tau and Sigma Functions on Abel-Jacobi Images

Atsushi Nakayashiki*
Department of Mathematics,
Tsuda College

Keijiro Yori

Dedicated to Michio Jimbo on his sixtieth birthday

Abstract

We study derivatives of Schur and tau functions from the view point of the Abel-Jacobi map. We apply the results to establish several properties of derivatives of the sigma function of an (n, s) curve. As byproducts we have an expression of the prime form in terms of derivatives of the sigma function and addition formulae which generalize those of Onishi for hyperelliptic sigma functions.

*e-mail: atsushi@tsuda.ac.jp

1 Introduction

The Riemann's theta function of an algebraic curve X of genus g can be considered, through the Abel-Jacobi map, as a multivalued multiplicative analytic function on X^g . The Riemann's vanishing theorem tells that the theta function shifted by the Riemann's constant vanishes identically on X^{g-1} . However it is possible to find certain derivatives of the theta function such that they become multivalued multiplicative analytic functions on X^{g-1} . Onishi [13] found such derivatives explicitly in the case of hyperelliptic curves. The extension of the results to the curve $y^n = f(x)$ is given in [9]. These explicit derivatives of the theta function are used to construct certain addition formulae in [13]. The aim of this paper is to generalize and clarify the structure of the results on derivatives and addition formulae in [13] by studying Schur and tau functions.

We consider a certain plane algebraic curve X , called an (n, s) curve [2], which contains curves $y^n = f(x)$ as a special case. As in [13] we study sigma functions [1, 11] rather than Riemann's theta function since it is simpler to describe derivatives. Sigma functions can be expressed by the tau function of the KP-hierarchy [4, 5, 12]. The expansion of the tau function with respect to Schur functions is known very explicitly due to Sato's theory of universal Grassmann manifold (UGM) [15, 14]. In the case corresponding to the sigma function of an (n, s) curve the expansion of the tau function begins from the Schur function $s_\lambda(t)$ corresponding to the partition λ determined from the gap sequence at ∞ of X . Notice that Schur functions themselves can be considered as a special case of tau functions.

For a theta function solution of the KP-hierarchy the image of the Abel-Jacobi map of a point on a Riemann surface is transformed, in the tau function, to the vector of the form

$$[z] = (z, z^2/2, z^3/3, \dots), \quad (1)$$

where z being a local coordinate at a base point. Being motivated by this we consider, in general, the map $z \mapsto [z]$ as an analogue of the Abel-Jacobi map for Schur and tau functions. For the Schur function corresponding to an (n, s) curve a similar map is considered in [2] as the rational limit of the Abel-Jacobi map.

The Schur function $s_\lambda(t)$, $t = (t_1, t_2, \dots)$, corresponding to a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ is the polynomial in t_1, t_2, \dots defined by

$$s_\lambda(t) = \det (p_{\lambda_i - i + j}(t))_{1 \leq i, j \leq l}, \quad \exp\left(\sum_{i=1}^{\infty} t_i k^i\right) = \sum_{i=1}^{\infty} p_i(t) k^i.$$

We firstly study, for each k satisfying $k \leq g$, the condition under which a derivative

$$\partial^\alpha s_\lambda([z_1] + \dots + [z_k]), \quad (2)$$

vanishes identically, where, for $\alpha = (\alpha_1, \alpha_2, \dots)$, ∂^α denote $\partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots$ and $\partial_i = \partial/\partial t_i$. A sufficient condition can easily be found. Let us define the weight of α by $\text{wt } \alpha = \sum_{i=1}^{\infty} i \alpha_i$ and set $N_{\lambda, k} = \lambda_{k+1} + \dots + \lambda_l$. Then the derivative (2) vanishes, if $\text{wt } \alpha < N_{\lambda, k}$.

Concerning to derivatives such that (2) does not vanish identically we have found two kinds of α satisfying $\text{wt } \alpha = N_{\lambda,k}$. One is $\alpha = (N_{\lambda,k}, 0, 0, \dots)$ for which the following recursive relation holds:

$$\partial_1^{N_{\lambda,k}} s_\lambda \left(\sum_{i=1}^k [z_i] \right) = \frac{c'_{\lambda,k}}{c'_{\lambda,k-1}} \partial_1^{N_{\lambda,k-1}} s_\lambda \left(\sum_{i=1}^{k-1} [z_i] \right) z_k^{\lambda_k} + O(z_k^{\lambda_k+1}), \quad (3)$$

where $c'_{\lambda,k}$ is a certain constant (Theorem 4).

The other kind of derivatives exist only for λ corresponding to a gap sequence. A gap sequence of genus g is a sequence of positive integers $w_1 < \dots < w_g$ such that its complement in the set of non-negative integers $\mathbb{Z}_{\geq 0}$ is a semi-group. To each gap sequence a partition $\lambda = (\lambda_1, \dots, \lambda_g)$ is associated by

$$\lambda = (w_g, \dots, w_2, w_1) - (g-1, \dots, 1, 0).$$

Let $w_1^* < w_2^* < \dots$ be the complement of $\{w_i\}$ in $\mathbb{Z}_{\geq 0}$. For each k the number m_k and the sequence $a_j^{(k)}$, $1 \leq j \leq m_k$, are defined by

$$\begin{aligned} m_k &= \#\{i | w_i^* < g - k\}, \\ (a_1^{(k)}, \dots, a_{m_k}^{(k)}) &= (w_{g-k}, w_{g-k-1}, \dots, w_{g-k-m_k+1}) - (w_1^*, \dots, w_{m_k}^*). \end{aligned}$$

Then $\sum_{j=1}^{m_k} a_j^{(k)} = N_{\lambda,k}$ and the following relation is valid:

$$\partial_{a_1^{(k)}} \cdots \partial_{a_{m_k}^{(k)}} s_\lambda \left(\sum_{i=1}^k [z_i] \right) = \pm \partial_{a_1^{(k-1)}} \cdots \partial_{a_{m_k}^{(k-1)}} s_\lambda \left(\sum_{i=1}^{k-1} [z_i] \right) z_k^{\lambda_k} + O(z_k^{\lambda_k+1}). \quad (4)$$

These derivatives generalize those of [13][9]. Our construction here clarifies the condition under which extensions of derivatives in [13] exist.

The tau function corresponding to a point of the cell UGM^λ of UGM specified by a partition λ has the expansion of the form

$$\tau(t) = s_\lambda(t) + \sum_{\lambda < \mu} \xi_\mu s_\mu(t). \quad (5)$$

We show that the vanishing property and the equations (3), (4) for Schur functions hold without any change if Schur functions are replaced by tau functions. To this end we need to study derivatives of Schur functions $s_\mu(t)$ corresponding to partitions μ satisfying $\lambda \leq \mu$ simultaneously. For example we have to study properties of " $a_j^{(k)}$ -derivatives" of $s_\mu(t)$ where $a_j^{(k)}$ are determined from λ .

In the case corresponding to (n, s) curves all the properties of tau functions established in this way are transplanted to sigma functions without much difficulty using the relation of the sigma function with the tau function.

For applications to addition formulae we need to study derivatives of Schur functions not only at $[z_1] + \dots + [z_k]$ but at $[z_1] - [z_2]$. In this case we have

$$\partial_1^{N_{\lambda,1}} s_\lambda([z_1] - [z_2]) = (-1)^{l-1} \frac{c_\lambda}{c'_{\lambda,1}} \partial_1^{N_{\lambda,1}} s_\lambda([z_1]) z_2^{l-1} + O(z_2^l), \quad (6)$$

where $N'_{\lambda,1} = \lambda_2 + \dots + \lambda_l - l + 1$ and c_λ is the constant given in Theorem 2. It can be proved using the rational analogue of the Riemann's vanishing theorem for Schur functions [2]. Again (6) and related properties are valid for tau and sigma functions without any change. As a corollary we obtain the expression of the prime form [6, 10, 11] in terms of a certain derivative of the sigma function and consequently closed addition formulae for sigma functions. Here "closed" means "without using prime form". The simplest example of the addition formula in the case of an (n, s) curve $X : y^n - \sum \lambda_{ij} x^i y^j = 0$, is

$$\frac{\partial_{u_1}^{N_{\lambda,2}} \sigma(p_2 + p_1) \partial_{u_1}^{N'_{\lambda,1}} \sigma(p_2 - p_1)}{(\partial_{u_1}^{N_{\lambda,1}} \sigma(p_1))^2 (\partial_{u_1}^{N_{\lambda,1}} \sigma(p_2))^2} = (-1)^g c_\lambda (c'_{\lambda,1})^{-4} c'_{\lambda,2} (x_2 - x_1), \quad (7)$$

where $p_i \in X$ is identified with its Abel-Jacobi image, $x_i = x(p_i)$ and λ is the partition corresponding to the gap sequence at ∞ of X . It generalizes the famous addition formula for Weierstrass' sigma function

$$\frac{\sigma(u_1 + u_2) \sigma(u_1 - u_2)}{\sigma(u_1)^2 \sigma(u_2)^2} = \wp(u_2) - \wp(u_1),$$

since $(x_i, y_i) = (\wp(u_i), \wp'(u_i))$, $i = 1, 2$, are two points on $y^2 = 4x^3 - g_2x - g_3$ and the right hand side can be written as $x_2 - x_1$. The formulae in [13] for hyperelliptic sigma functions are recovered if we use " $a_j^{(k)}$ -derivatives" instead of u_1 -derivative (see the remark after Corollary 10).

The present paper is organized as follows. In section two properties of derivatives of Schur functions are studied. The notion of gap sequence and the sequence $a_i^{(k)}$ are introduced. We lift the properties of Schur function in section two to functions satisfying similar expansion to the tau functions of the KP-hierarchy in section 3. In section 4 the properties on derivatives of the sigma function are proved using the sigma function expression of the tau function. The expression of the prime form in terms of a derivative of the sigma function of an (n, s) curve is given in section 5. Addition formulae for sigma functions are proved.

2 Schur function

A sequence of non-negative integers $\lambda = (\lambda_1, \dots, \lambda_l)$ satisfying $\lambda_1 \geq \dots \geq \lambda_l$ is called a partition. The number of non-zero elements in λ is called the length of λ and is denoted by $l(\lambda)$. We identify λ with partitions which are obtained from λ by adding arbitrary number of 0's, i.e. $(\lambda_1, \dots, \lambda_l, 0, \dots, 0)$. We set $|\lambda| = \lambda_1 + \dots + \lambda_l$.

Let $t = (t_1, t_2, t_3, \dots)$ and $p_n(t)$ the polynomial in t defined by

$$\exp\left(\sum_{n=1}^{\infty} t_n k^n\right) = \sum_{n=0}^{\infty} p_n(t) k^n. \quad (8)$$

We set $p_n(t) = 0$ for $n < 0$.

For a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ Schur functions $s_\lambda(t)$ and $S_\lambda(x)$ are defined by

$$\begin{aligned} s_\lambda(t) &= \det(p_{\lambda_i - i + j}(t))_{1 \leq i, j \leq l}, \\ S_\lambda(x) &= \frac{\det(x_j^{\lambda_i + l - i})_{1 \leq i, j \leq l}}{\prod_{i < j} (x_i - x_j)}. \end{aligned} \quad (9)$$

The function $S_\lambda(x)$ is a symmetric polynomial of x_1, \dots, x_l which is homogeneous of degree $|\lambda|$.

We introduce the symbol $[x]$ by

$$[x] = (x, \frac{x^2}{2}, \frac{x^3}{3}, \dots),$$

which is an analogue of Abel-Jacobi map in the theory of Schur functions. With this symbol, $s_\lambda(t)$ and $S_\lambda(x)$ are related by

$$s_\lambda\left(\sum_{i=1}^n [x_i]\right) = S_\lambda(x),$$

for $n \geq l(\lambda)$. From this relation we have

Proposition 1 *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition of length l . Then*

$$(i) \ s_\lambda\left(\sum_{i=1}^l [x_i]\right) = s_{(\lambda_1, \dots, \lambda_{l-1})}\left(\sum_{i=1}^{l-1} [x_i]\right) x_l^{\lambda_l} + O(x_l^{\lambda_l+1}).$$

$$(ii) \text{ If } k < l, \ s_\lambda\left(\sum_{i=1}^k [x_i]\right) = 0.$$

Proof. (i) It immediately follows from the definition of $S_\lambda(x)$.

(ii) We have

$$s_\lambda\left(\sum_{i=1}^k [x_i]\right) = s_\lambda\left(\sum_{i=1}^k [x_i] + [0] + \dots + [0]\right).$$

The right hand side is zero by (i) since $\lambda_l \geq 1$. ■

Let G^c be a subset of the set of non-negative integers $\mathbb{Z}_{\geq 0}$. We assume that G^c is a semi-group, that is, it is closed under addition and contains 0. Set $G = \mathbb{Z}_{\geq 0} \setminus G^c$.

Definition 1 *Let g be a positive integer. G is called a gap sequence of genus g , if $\#G = g$. Elements of G and G^c are called gaps and non-gaps respectively.*

For a gap sequence of genus g enumerate elements of G and G^c respectively as

$$\begin{aligned} w_1 &< w_2 < \dots < w_g, \\ 0 &= w_1^* < w_2^* < w_3^* < \dots. \end{aligned}$$

Then $w_1 = 1$. For, otherwise G^c contains 1 and $G^c = \mathbb{Z}_{\geq 0}$ which is impossible due to $g \geq 1$. With this notation in mind we sometimes use (w_1, \dots, w_g) to denote a gap sequence instead of $\{w_1, \dots, w_g\}$.

Example 1 Let (n, s) be a pair of relatively prime integers such that $n, s \geq 2$. We set

$$G^c = \{in + js \mid i, j \geq 0\}.$$

Then G is a gap sequence of genus $g = 1/2(n-1)(s-1)$ [2]. We call G the gap sequence of type (n, s) . It is characterized by the condition that G^c is generated by two elements.

Example 2 Let $G = \{1, 2, 3, 7\}$ and $G^c = \mathbb{Z}_{\geq 0} \setminus G$. Then G is a gap sequence of genus four. In this case G^c is generated by 4, 5, 6. Therefore G is not of type (n, s) for any (n, s) .

In this way the gap sequences are classified by the minimum number of generators of G^c .

For a gap sequence $\{w_1, \dots, w_g\}$ we associate a partition λ by

$$\lambda = (w_g, \dots, w_1) - (g-1, \dots, 1, 0).$$

A special property of the partition determined from a gap sequence is the following.

Proposition 2 *If λ is determined from a gap sequence (w_1, \dots, w_g) , then $s_\lambda(t)$ does not depend on t_i , $i \notin \{w_1, \dots, w_g\}$.*

In order to prove the proposition we introduce some notation.

For a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ we associate a strictly decreasing sequence of numbers \bar{w}_i by

$$(\bar{w}_1, \dots, \bar{w}_l) = (\lambda_1, \dots, \lambda_l) + (l-1, l-2, \dots, 0).$$

By this correspondence the set of partitions of length at most l bijectively corresponds to the set of strictly decreasing sequence non-negative integers $\bar{w}_1 > \dots > \bar{w}_l \geq 0$.

For $(\bar{w}_1, \dots, \bar{w}_l)$ we set

$$(w_l, \dots, w_1) = (\bar{w}_1, \dots, \bar{w}_l).$$

The introduction of the notation \bar{w}_i is for the sake of simplicity in proofs and that of w_i is for the sake of being consistent with the notation of gap sequence.

For integers i_1, \dots, i_l define the symbol $[i_1, \dots, i_l]$ as the determinant of the $l \times l$ matrix whose j -th row is

$$(\dots, p_{i_j-1}(t), p_{i_j}(t)).$$

We write $[i_1, \dots, i_l](t)$ if it is necessary to write explicitly the dependence on t .

By the definition, $[i_1, \dots, i_l]$ is skew symmetric in the numbers i_1, \dots, i_l and becomes zero if two numbers coincide or some number is negative.

With this notation

$$s_\lambda(t) = [\bar{w}_1, \dots, \bar{w}_l].$$

Differentiating (8) by t_i we have

$$\partial_i p_n(t) = p_{n-i}(t), \quad \partial_i = \frac{\partial}{\partial t_i}.$$

Therefore we have

$$\partial_i s_\lambda(t) = \sum_{j=1}^l [\bar{w}_1, \dots, \bar{w}_j - i, \dots, \bar{w}_l].$$

Proof of Proposition 2.

We have to show, for $i \geq 1$,

$$\partial_{w_i^*} s_\lambda(t) = \sum_{j=1}^l D_j = 0, \quad D_j = [w_l, \dots, w_j - w_i^*, \dots, w_1]. \quad (10)$$

If $w_j - w_i^* < 0$, obviously $D_j = 0$. Suppose that $w_j - w_i^* > 0$. Let $G = \{w_1, \dots, w_g\}$. Then $w_j - w_i^* \in G$. For, if $w_j - w_i^* \in G^c$ then $w_j \in G^c + w_i^* \subset G^c$ which is absurd. Thus $w_j - w_i^* = w_k$ for some k . Notice that $w_i^* \geq 1$ and $k \neq j$, since $i \geq 1$. Therefore $D_j = 0$ because two rows coincide. Consequently (10) is proved. ■

Definition 2 Let G be a gap sequence of genus g . For $0 \leq k \leq g-1$ we define a positive integer m_k and a sequence of integers $a_i^{(k)}$, $1 \leq i \leq m_k$ by

$$\begin{aligned} m_k &= \#\{i \mid w_i^* < g - k\}, \\ (a_1^{(k)}, \dots, a_{m_k}^{(k)}) &= (w_{g-k}, w_{g-k-1}, \dots, w_{g-k-m_k+1}) - (w_1^*, \dots, w_{m_k}^*). \end{aligned}$$

Example 3 For the gap sequence of type $(2, 2g+1)$ we have

$$(w_1, w_2, \dots, w_g) = (1, 3, \dots, 2g-1), \quad (w_1^*, w_2^*, w_3^*, \dots) = (0, 2, 4, \dots).$$

Then

$$\begin{aligned} m_k &= \#\{i \mid 2i - 2 < g - k\} = \left\lfloor \frac{g - k + 1}{2} \right\rfloor, \\ (a_1^{(k)}, a_2^{(k)}, \dots) &= (2g - 2k - 1, 2g - 2k - 5, 2g - 2k - 9, \dots). \end{aligned}$$

This sequence recovers the rule for derivatives in [13].

For a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ and a number k such that $0 \leq k \leq l-1$ we set

$$N_{\lambda,k} = \lambda_{k+1} + \dots + \lambda_l. \quad (11)$$

Lemma 1 (i) $a_1^{(k)} > \dots > a_{m_k}^{(k)} \geq 1$.

(ii) Each $a_i^{(k)}$ belongs to G .

(iii) Let λ be the partition determined from G then

$$\sum_{i=1}^{m_k} a_i^{(k)} = N_{\lambda, k}.$$

Proof. (i) Notice that $(w_{g-k}, w_{g-k-1}, \dots)$ is strictly decreasing and (w_1^*, w_2^*, \dots) is increasing. Therefore $\{a_i^{(k)}\}$ is strictly decreasing. Since G and G^c are complement to each other we have

$$\{0, 1, \dots, g-k-1\} = \{w_1^*, \dots, w_{m_k}^*\} \sqcup \{w_1, \dots, w_{g-k-m_k}\}. \quad (12)$$

Then, by the definition of the number m_k ,

$$\begin{aligned} w_1^* &< \dots < w_{m_k}^* < g-k \leq w_{m_k+1}^* < \dots, \\ w_1 &< \dots < w_{g-k-m_k} < g-k \leq w_{g-k-m_k+1} < \dots < w_{g-k} < \dots. \end{aligned} \quad (13)$$

In particular $a_{m_k}^{(k)} = w_{g-k-m_k+1} - w_{m_k}^* \geq 1$.

(ii) Suppose that $a_j^{(k)} \in G^c$. Since G^c is a semi-group we have

$$w_{g-k-j+1} = a_j^{(k)} + w_j^* \in G^c,$$

which is absurd. Thus $a_j^{(k)} \in G$.

(iii) By (12) we have

$$\begin{aligned} \sum_{i=1}^{m_k} a_i^{(k)} &= \sum_{i=g-k-m_k+1}^{g-k} w_i - \sum_{i=1}^{m_k} w_i^* \\ &= \sum_{i=g-k-m_k+1}^{g-k} w_i - \left(\sum_{i=1}^{g-k-1} i - \sum_{i=1}^{g-k-m_k} w_i \right) \\ &= \sum_{i=1}^{g-k} w_i - \sum_{i=1}^{g-k-1} i = \sum_{i=k+1}^g \lambda_i. \end{aligned}$$

■

For $\alpha = (\alpha_1, \alpha_2, \dots)$ with finite number of non-zero components we define the weight of α and the symbol ∂^α by

$$\text{wt } \alpha = \sum_{i=1}^{\infty} i \alpha_i, \quad \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots$$

The weight of ∂^α is defined to be the weight of α .

Proposition 3 Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition and $0 \leq k \leq l-1$. If $\text{wt } \alpha < N_{\lambda,k}$ we have

$$\partial^\alpha s_\lambda \left(\sum_{i=1}^k [x_i] \right) = 0.$$

For $k = 0$ the right hand side should be understood as $\partial^\alpha s_\lambda(0)$.

Proof. Notice that $\partial^\alpha s_\lambda(t)$ is a linear combination of determinants of the form

$$[\bar{w}_1 - r_1, \dots, \bar{w}_l - r_l], \quad r_1 + \dots + r_l = \text{wt } \alpha. \quad (14)$$

If (14) is not zero, $\bar{w}_i - r_i$ are all non-negative and different. Thus there exists a permutation (i_1, \dots, i_l) of $(1, \dots, l)$ such that

$$\bar{w}_{i_1} - r_{i_1} > \dots > \bar{w}_{i_l} - r_{i_l} \geq 0.$$

Let μ be the partition corresponding to this strictly decreasing sequence. Then

$$s_\mu(t) = [\bar{w}_{i_1} - r_{i_1}, \dots, \bar{w}_{i_l} - r_{i_l}].$$

If $l(\mu) > k$, $s_\mu \left(\sum_{i=1}^k [x_i] \right) = 0$ by (ii) of Proposition 1.

We prove that $l(\mu) \leq k$ is impossible if $\text{wt } \alpha < N_{\lambda,k}$. Suppose that $l(\mu) \leq k$. Then $\mu = (\mu_1, \dots, \mu_k, 0, \dots, 0)$ and

$$\bar{w}_{i_l} - r_{i_l} = 0, \quad \bar{w}_{i_{l-1}} - r_{i_{l-1}} = 1, \dots, \quad \bar{w}_{i_{k+1}} - r_{i_{k+1}} = l - k - 1.$$

Therefore

$$r_{i_l} = \bar{w}_{i_l}, \quad r_{i_{l-1}} = \bar{w}_{i_{l-1}} - 1, \dots, \quad r_{i_{k+1}} = \bar{w}_{i_{k+1}} - (l - k - 1),$$

and we have

$$\begin{aligned} r_{i_l} + \dots + r_{i_{k+1}} &= \bar{w}_{i_l} + \dots + \bar{w}_{i_{k+1}} - (1 + 2 + \dots + l - k - 1) \\ &\geq \bar{w}_l + \dots + \bar{w}_{k+1} - (1 + 2 + \dots + l - k - 1). \end{aligned}$$

On the other hand

$$\begin{aligned} r_{i_l} + \dots + r_{i_{k+1}} &\leq r_l + \dots + r_1 = \text{wt } \alpha \\ &< \lambda_{k+1} + \dots + \lambda_l \\ &= \bar{w}_{k+1} + \dots + \bar{w}_l - (1 + 2 + \dots + l - k - 1), \end{aligned}$$

which is a contradiction. Thus Proposition 3 is proved. ■

Theorem 1 Let $\lambda = (\lambda_1, \dots, \lambda_g)$ be the partition determined from a gap sequence of genus g , $0 \leq k \leq g$ and $a_j^{(k)}$ the associated sequence of numbers for $k \neq g$. We set $s_{(\lambda_1, \dots, \lambda_k)}(\sum_{i=1}^k [x_i]) = 1$ for $k = 0$ and $\partial_{a_1^{(k)}} \cdots \partial_{a_{m_k}^{(k)}} = 1$ for $k = g$.

(i) We have

$$\partial_{a_1^{(k)}} \cdots \partial_{a_{m_k}^{(k)}} s_{\lambda} \left(\sum_{i=1}^k [x_i] \right) = c_k s_{(\lambda_1, \dots, \lambda_k)} \left(\sum_{i=1}^k [x_i] \right),$$

where $c_k = \pm 1$, $k \neq g$ is given by the sign of the permutation

$$c_k = \text{sgn} \begin{pmatrix} w_1^* & \dots & w_{m_k}^* & w_{g-k-m_k} & \dots & w_1 \\ g-k-1 & g-k-2 & \dots & \dots & 1 & 0 \end{pmatrix},$$

and $c_g = 1$.

(ii) Let $\mu = (\mu_1, \dots, \mu_g)$ be a partition such that $\mu_i = \lambda_i$ for $k+1 \leq i \leq g$. Then

$$\partial_{a_1^{(k)}} \cdots \partial_{a_{m_k}^{(k)}} s_{\mu} \left(\sum_{i=1}^k [x_i] \right) = c_k s_{(\mu_1, \dots, \mu_k)} \left(\sum_{i=1}^k [x_i] \right),$$

where c_k is the same as in (i).

Remark 1 For the gap sequence of type (n, s) it can be checked that the derivative determined from the sequence $a_j^{(k)}$ is the same as that found in [9]. In that case (i) of Theorem 1 is proved in that paper.

Lemma 2 Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition, $0 \leq k \leq l-1$ and r_1, \dots, r_l non-negative integers. Suppose that the following conditions:

$$\sum_{i=1}^l r_i = N_{\lambda, k}, \tag{15}$$

$$[\bar{w}_1 - r_1, \dots, \bar{w}_l - r_l] \left(\sum_{i=1}^k [x_i] \right) \neq 0. \tag{16}$$

Then

(i) We have $r_i = 0$ for $1 \leq i \leq k$.

(ii) The sequence $(\bar{w}_{k+1} - r_{k+1}, \dots, \bar{w}_l - r_l)$ is a permutation of $(l-k-1, \dots, 1, 0)$.

(iii) We have

$$[\bar{w}_1 - r_1, \dots, \bar{w}_l - r_l] \left(\sum_{i=1}^k [x_i] \right) = c s_{(\lambda_1, \dots, \lambda_k)} \left(\sum_{i=1}^k [x_i] \right),$$

where $c = \pm 1$.

Proof. By the assumption (16) there exists a permutation (i_1, \dots, i_l) of $(1, \dots, l)$ and a partition $\mu = (\mu_1, \dots, \mu_l)$ such that

$$\begin{aligned}\bar{w}_{i_1} - r_{i_1} &> \dots > \bar{w}_{i_l} - r_{i_l} \geq 0, \\ s_\mu(t) &= [\bar{w}_{i_1} - r_{i_1}, \dots, \bar{w}_{i_l} - r_{i_l}],\end{aligned}$$

and $l(\mu) \leq k$ as in the proof of Proposition 3. In particular $\mu_i = 0$ for $i \geq k+1$ which means

$$\bar{w}_{i_l} - r_{i_l} = 0, \quad \dots, \quad \bar{w}_{i_{k+1}} - r_{i_{k+1}} = l - k - 1.$$

By a similar calculation to that in the proof of Proposition 3 we have

$$\begin{aligned}r_{i_{k+1}} + \dots + r_{i_l} &= \bar{w}_{i_{k+1}} + \dots + \bar{w}_{i_l} - (1 + 2 + \dots + l - k - 1) \\ &\geq \bar{w}_{k+1} + \dots + \bar{w}_l - (1 + 2 + \dots + l - k - 1),\end{aligned}\tag{17}$$

and

$$\begin{aligned}r_{i_{k+1}} + \dots + r_{i_l} &\leq r_1 + \dots + r_l \\ &= \lambda_{k+1} + \dots + \lambda_l \\ &= \bar{w}_{k+1} + \dots + \bar{w}_l - (1 + 2 + \dots + l - k - 1),\end{aligned}\tag{18}$$

where we use (15). Therefore every inequalities in (17) and (18) are equalities. Then $r_{i_1} = \dots = r_{i_k} = 0$ by (18) and (i_{k+1}, \dots, i_l) is a permutation of $(k+1, \dots, l)$ by (17). It, then, implies that (i_1, \dots, i_k) is a permutation of $(1, \dots, k)$.

Since

$$(\bar{w}_{i_1} - r_{i_1}, \dots, \bar{w}_{i_l} - r_{i_l}) = (\bar{w}_{i_1}, \dots, \bar{w}_{i_k}, \bar{w}_{i_{k+1}} - r_{i_{k+1}}, \dots, \bar{w}_{i_l} - r_{i_l})$$

and it is strictly decreasing, $(i_1, \dots, i_k) = (1, \dots, k)$. Thus

$$[\bar{w}_{i_1} - r_{i_1}, \dots, \bar{w}_{i_l} - r_{i_l}] = [\bar{w}_1, \dots, \bar{w}_k, l - k - 1, \dots, 1, 0].\tag{19}$$

Lemma 3 *For a positive integer m and a set of integers i_1, \dots, i_k we have*

$$[i_1, \dots, i_k, m - 1, \dots, 1, 0] = [i_1 - m, \dots, i_k - m].$$

Proof. Expand the determinant at $m+k$ -th row, $m+k-1$ -th row, ..., until $k+1$ -st row successively and get the result. ■

Applying the lemma to (19) we have

$$\begin{aligned}[\bar{w}_{i_1} - r_{i_1}, \dots, \bar{w}_{i_l} - r_{i_l}] &= [\bar{w}_1 - (l - k), \dots, \bar{w}_k - (l - k)] \\ &= s_{(\lambda_1, \dots, \lambda_k)}(t).\end{aligned}$$

Since (i_1, \dots, i_l) is a permutation of $(1, \dots, l)$,

$$[\bar{w}_1 - r_1, \dots, \bar{w}_l - r_l] = \pm s_{(\lambda_1, \dots, \lambda_k)}(t).$$

■

Proof of Theorem 1.

In this proof we fix k and denote $a_j^{(k)}$ simply by a_j . Recall that

$$s_\lambda(t) = [w_g, \dots, w_1].$$

We compute the value of $\partial_{a_1} \cdots \partial_{a_{m_k}} s_\lambda(t)$ at $t = t^{(k)} := [x_1] + \cdots + [x_k]$.

Step 1. We first consider the term for which the row labeled by $w_{g-k-(i-1)}$ is differentiated by ∂_{a_i} for $1 \leq i \leq m_k$. It is of the form

$$A := [w_g, \dots, w_{g-k+1}, w_{g-k} - a_1, \dots, w_{g-k-(m_k-1)} - a_{m_k}, w_{g-k-m_k}, \dots, w_1].$$

By the definition of a_i

$$w_{g-k-(i-1)} - a_i = w_i^*.$$

Therefore

$$A = [w_g, \dots, w_{g-k+1}, w_1^*, \dots, w_{m_k}^*, w_{g-k-m_k}, \dots, w_1].$$

Using (12) we have

$$\begin{aligned} A &= c_k[w_g, \dots, w_{g-k+1}, g-k-1, \dots, 1, 0] \\ &= c_k s_{(\lambda_1, \dots, \lambda_k)}(t). \end{aligned}$$

Step 2. We prove that the terms differentiated in a different way from that in Step 1 are zero at $t = t^{(k)}$.

By Lemma 1 (iii) and Lemma 2 (i) the term is zero at $t^{(k)}$ if some row corresponding to w_i , $g-k+1 \leq i \leq g$, is differentiated. Therefore, for non-zero terms, only the last $g-k$ rows are differentiated.

So let us consider a term for which only some of last $g-k$ rows are differentiated. Notice that a term is zero if some row is differentiated more than once. In fact some row corresponding to w_j with $g-k-m_k+1 \leq j \leq g-k$ is not differentiated in this case. By (13) $w_j \geq g-k$. Consequently it is impossible for the sequence (w_{g-k}, \dots, w_1) to be a permutation of $(g-k-1, \dots, 1, 0)$. Then this term is zero at $t^{(k)}$ by Lemma 2 (ii).

As a consequence of the above argument we know that a term is zero if some row labeled by w_j with $g-k-m_k+1 \leq j \leq g-k$ is not differentiated. So let us consider a term for which each row corresponding to w_j with $g-k-m_k+1 \leq j \leq g-k$ is differentiated exactly once. We assume that the row corresponding $w_{g-k-(i-1)}$ is differentiated by ∂_{a_i} for $1 \leq i < j$ with some $j \leq m_k$ and ∂_{a_j} differentiates the row corresponding to $w_{g-k-(j'-1)}$ for some j' with $j < j'$. We have

$$w_{g-k} - a_1 = w_1^*, \quad \dots, \quad w_{g-k-(j-2)} - a_{j-1} = w_{j-1}^*,$$

and

$$\begin{aligned} w_{g-k-(j'-1)} - a_j &= w_{g-k-(j'-1)} - (w_{g-k-(j-1)} - w_j^*) \\ &= w_j^* - (w_{g-k-(j-1)} - w_{g-k-(j'-1)}) < w_j^*. \end{aligned} \quad (20)$$

If $w_{g-k-(j'-1)} - a_j$ belongs to G^c , we have

$$w_{g-k-(j'-1)} - a_j \in \{w_1^*, \dots, w_{j-1}^*\},$$

by (20). Thus the term is zero since two rows coincide.

Suppose that $w_{g-k-(j'-1)} - a_j$ belongs to G . Then

$$w_{g-k-(j'-1)} - a_j \in \{w_1, \dots, w_{g-k-m_k}\},$$

since $w_j^* < g - k$ and (13). In this case the term in consideration is zero since again two rows coincide. Thus (i) of Theorem 1 is proved.

Step 3. We prove (ii) of Theorem 1. Let $w'_g > \dots > w'_1$ be the strictly decreasing sequence corresponding to μ , that is,

$$(w'_g, \dots, w'_1) = (\mu_1, \dots, \mu_g) + (g-1, \dots, 1, 0).$$

By assumption $w_i = w'_i$ for $1 \leq i \leq g - k$. Define $w_i'^*$, $i \geq 0$ by

$$\begin{aligned} \{w_i'^* \mid i \geq 0\} &= \mathbb{Z}_{\geq 0} \setminus \{w_i'\}, \\ 0 &= w_1'^* < w_2'^* < \dots. \end{aligned}$$

Then $w_i^* = w_i'^*$ for $1 \leq i \leq m_k$, since

$$\begin{aligned} \{w_1^*, \dots, w_{m_k}^*\} \sqcup \{w'_1, \dots, w'_{g-k-m_k}\} &= \{w_1^*, \dots, w_{m_k}^*\} \sqcup \{w_1, \dots, w_{g-k-m_k}\} \\ &= \{0, 1, \dots, g - k - 1\}. \end{aligned}$$

As a consequence the arguments in step 1 and step 2 are valid without any change if w_i , w_i^* are replaced by w'_i , $w_i'^*$ respectively. ■

Next we study properties of Schur functions with respect to t_1 derivative.

Theorem 2 *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition, (w_l, \dots, w_1) the corresponding strictly decreasing sequence and $0 \leq k \leq l$. Then*

$$\partial_1^{N_{\lambda,k}} s_{\lambda} \left(\sum_{i=1}^k [x_i] \right) = c'_{\lambda,k} s_{(\lambda_1, \dots, \lambda_k)} \left(\sum_{i=1}^k [x_i] \right),$$

where

$$c'_{\lambda,k} = \frac{N_{\lambda,k}!}{\prod_{i=1}^{l-k} w_i!} \prod_{i < j}^{l-k} (w_j - w_i).$$

Proof. We have

$$s_\lambda(t) = [w_l, \dots, w_1].$$

By Leibniz's rule

$$\partial_1^{N_{\lambda,k}} s_\lambda(t) = \sum_{r_1 + \dots + r_l = N_{\lambda,k}} \frac{N_{\lambda,k}!}{r_1! \dots r_l!} [w_l - r_l, \dots, w_1 - r_1]. \quad (21)$$

By Lemma 2, if $[w_l - r_l, \dots, w_1 - r_1](t^{(k)}) \neq 0$ then $r_i = 0$ for $l - k + 1 \leq i \leq l$, $(w_{l-k}, \dots, w_1 - r_1)$ is a permutation of $(l - k - 1, \dots, 1, 0)$ and

$$[w_l - r_l, \dots, w_1 - r_1](t^{(k)}) = \text{sgn} \begin{pmatrix} w_{l-k} & \dots & \dots & w_1 - r_1 \\ l - k - 1 & \dots & 1 & 0 \end{pmatrix}.$$

In this case we can write

$$w_i - r_i = \sigma(i - 1), \quad 1 \leq i \leq l - k,$$

for some σ of an element of the symmetric group S_{l-k} acting on $\{0, 1, \dots, l - k - 1\}$. We define $1/n! = 0$ for $n < 0$ for the sake of convenience. Then

$$\partial_1^{N_{\lambda,k}} s_\lambda(t^{(k)}) = A_{\lambda,k} s_{(\lambda_1, \dots, \lambda_k)}(t^{(k)}),$$

where

$$A_{\lambda,k} = \sum_{\sigma \in S_{l-k}} \text{sgn } \sigma \frac{N_{\lambda,k}!}{(w_1 - \sigma(0))! \dots (w_{l-k} - \sigma(l - k - 1))!}.$$

We have

$$\frac{A_{\lambda,k}}{N_{\lambda,k}!} = \det \left(\frac{1}{(w_i - (j - 1))!} \right)_{1 \leq i, j \leq l-k} \quad (22)$$

$$= \prod_{i=1}^{l-k} \frac{1}{w_i!} \det \left(\prod_{m=0}^{j-2} (w_i - m) \right)_{1 \leq i, j \leq l-k}, \quad (23)$$

where we set $\prod_{m=0}^{j-2} (w_i - m) = 1$ for $j = 1$. Notice that the rule $1/n! = 0$ for $n < 0$ is taken into account in rewriting (22) to (23), since, if $w_i - (j - 1) < 0$ then $\prod_{m=0}^{j-2} (w_i - m) = 0$.

Let us set

$$D = \det \left(\prod_{m=0}^{j-2} (w_i - m) \right)_{1 \leq i, j \leq l-k}.$$

Expanding $\prod_{m=0}^{j-2} (w_i - m)$ in w_i we easily have

$$D = \det (w_i^{j-1})_{1 \leq i, j \leq l-k} = \prod_{i < j}^{l-k} (w_j - w_i),$$

and consequently

$$\frac{A_{\lambda, k}}{N_{\lambda, k}!} = \frac{\prod_{i < j}^{l-k} (w_j - w_i)}{\prod_{i=1}^{l-k} w_i!}.$$

■

In order to study addition formulae of sigma functions we need to study properties of Schur functions at $t = [x_1] - [x_2]$.

For a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ let $\lambda' = (\lambda'_1, \dots, \lambda'_{l'})$ be the conjugate of λ , i.e. $\lambda'_i = \#\{j \mid \lambda_j \geq i\}$.

Theorem 3 *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition of length l , $\lambda' = (\lambda'_1, \dots, \lambda'_{l'})$ and $\tilde{\lambda}' = (\lambda'_1 - 1, \dots, \lambda'_{l'} - 1)$. Then*

$$s_\lambda([x] - \sum_{i=1}^{l'} [x_i]) = (-1)^{N_{\lambda, 1}} s_{\tilde{\lambda}'}(\sum_{i=1}^{l'} [x_i]) \prod_{j=1}^{l'} (x - x_j).$$

Proof. This theorem is essentially proved in the proof of Theorem 5.5 in [2]. In [2] λ is assumed to be the partition corresponding to the gap sequence of type (n, s) . In that case $\lambda = \lambda'$ and the assertion in this theorem is not stated. Here we give a proof since it is a key theorem for applications to addition formulae. For the notational simplicity we prove the assertion by interchanging λ and λ' . All facts and notation concerning Schur and symmetric functions used in this proof can be found in [8]

Let $e_i = e_i(x_1, \dots, x_m)$ be the elementary symmetric function:

$$\prod_{i=1}^m (t + x_i) = \sum_{i=0}^m e_i t^{m-i}. \quad (24)$$

They satisfy the relation

$$e_i(x_1, \dots, x_m) = e_i(x_1, \dots, x_{m-1}) + x_m e_{i-1}(x_1, \dots, x_{m-1}). \quad (25)$$

In general, for a partition $\mu = (\mu_1, \dots, \mu_m)$, the following equation holds:

$$S_{\mu'}(x_1, \dots, x_m) = \det(e_{\mu_i - i + j})_{1 \leq i, j \leq m}. \quad (26)$$

Let \mathbf{a}_j be the column vector defined by

$$\mathbf{a}_j = {}^t(e_{\lambda_1 - 1 + j}, e_{\lambda_2 - 2 + j}, \dots, e_{\lambda_l - l + j}),$$

where $e_r = e_r(x, x_1, \dots, x_l)$.

By (25), (26) we have

$$\begin{aligned}
s_{\lambda'}([x] + [x_1] + \dots + [x_l]) &= S_{\lambda'}(x, x_1, \dots, x_l) \\
&= \det(e_{\lambda_i - i + j})_{1 \leq i, j \leq l} \\
&= \det(\mathbf{a}_1 + x\mathbf{a}_0, \mathbf{a}_2 + x\mathbf{a}_1, \dots, \mathbf{a}_l + x\mathbf{a}_{l-1}) \\
&= \sum_{j=0}^l x^j \det(\mathbf{a}_0, \dots, \mathbf{a}_{j-1}, \mathbf{a}_{j+1}, \dots, \mathbf{a}_l) \\
&= \det \begin{pmatrix} 1 & -x & \cdots & (-x)^l \\ \mathbf{a}_0 & \mathbf{a}_1 & \cdots & \mathbf{a}_l \end{pmatrix}. \tag{27}
\end{aligned}$$

Let $p_r = \sum_{i=1}^l x_i^k$ be the power sum symmetric function, ω , $\widehat{\omega}$ and ι the automorphisms of the ring of symmetric polynomials in x_1, \dots, x_l defined by

$$\widehat{\omega}(p_r) = (-1)^r p_r, \quad \iota(p_r) = -p_r, \quad \omega = \iota \circ \widehat{\omega}. \tag{28}$$

Notice that $\widehat{\omega}$ is, in terms of x_j , the map sending x_j to $-x_j$ for $1 \leq j \leq l$. Then

$$s_{\lambda'}([x] - \sum_{i=1}^l [x_i]) = (-1)^{|\lambda|} \omega \left(s_{\lambda'}([-x] + \sum_{i=1}^l [x_i]) \right). \tag{29}$$

It can be checked by computing the right hand side using (28) and the relation $S_{\mu}(-x_1, \dots, -x_m) = (-1)^{|\mu|} S_{\mu}(x_1, \dots, x_m)$.

Let $h_i = h_i(x_1, \dots, x_l)$ be the complete symmetric function:

$$\frac{1}{\prod_{i=1}^l (1 - tx_i)} = \sum_{i=0}^{\infty} h_i x^i.$$

Then $\omega(e_i) = h_i$ and

$$\omega(\mathbf{a}_j) = {}^t(h_{\lambda_1 - 1 + j}, \dots, h_{\lambda_l - l + j}). \tag{30}$$

By (27) and (29) we have

$$s_{\lambda'}([x] - [x_1] - \dots - [x_l]) = (-1)^{|\lambda|} \det \begin{pmatrix} 1 & x & \cdots & x^l \\ \omega(\mathbf{a}_0) & \omega(\mathbf{a}_1) & \cdots & \omega(\mathbf{a}_l) \end{pmatrix}. \tag{31}$$

Using the relation,

$$\sum_{j=0}^k (-1)^j e_j h_{k-j} = 0, \quad k \geq 1, \tag{32}$$

we have

$$\sum_{j=0}^k (-1)^j e_j \omega(\mathbf{a}_{l-j}) = \mathbf{o}. \tag{33}$$

By (24), (30), (31), (33) we obtain

$$\begin{aligned} s_{\lambda'}([x] - \sum_{i=1}^l [x_i]) &= (-1)^{l+|\lambda|} \det(\omega(\mathbf{a}_0), \dots, \omega(\mathbf{a}_{l-1})) \prod_{j=1}^l (x - x_j) \\ &= (-1)^{N_{\lambda',1}} \det(h_{\lambda_i-1-i+j})_{1 \leq i,j \leq l} \prod_{j=1}^l (x - x_j). \end{aligned}$$

Then the theorem follows from

$$S_{(\mu_1, \dots, \mu_m)}(x_1, \dots, x_m) = \det(h_{\mu_i-i+j})_{1 \leq i,j \leq m}.$$

■

Corollary 1 *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition of length l . Then $s_\lambda([x_1] - [x_2])$ is not identically zero if and only if $\lambda_i = 1$ for $2 \leq i \leq l$, that is, λ is a hook.*

Proof. Setting $x_i = 0$ for $2 \leq i \leq l'$ in Theorem 3 we have

$$s_\lambda([x] - [x_1]) = (-1)^{N_{\lambda,1}} s_{\tilde{\lambda}'}([x_1]) x_1^{l'-1} (x - x_1). \quad (34)$$

Thus $s_\lambda([x] - [x_1]) \neq 0$ is equivalent to $s_{\tilde{\lambda}'}([x_1]) \neq 0$. The latter is equivalent to the condition that the length of $\tilde{\lambda}'$ is one. It means that $\lambda' = (\lambda'_1, 1^{l'-1})$ which is equivalent to that λ is a hook. ■

Theorem 4 *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition of length l , (w_l, \dots, w_1) the corresponding sequence and $N'_{\lambda,1} = \sum_{i=2}^l \lambda_i - l + 1$.*

(i) *If $n < N'_{\lambda,1}$*

$$\partial_1^n s_\lambda([x_1] - [x_2]) = 0.$$

(ii) *We have*

$$\partial_1^{N'_{\lambda,1}} s_\lambda([x_1] - [x_2]) = c_\lambda s_{(\lambda_1, 1^{l-1})}([x_1] - [x_2]),$$

where

$$c_\lambda = \frac{N'_{\lambda,1}!}{\prod_{i=1}^{l-1} (w_i - 1)!} \prod_{i < j}^{l-1} (w_j - w_i).$$

(iii) *Let $\mu = (\mu_1, \dots, \mu_{l'})$ be a partition of length $l' \geq l$ such that $\mu_i = \lambda_i$ for $2 \leq i \leq l$ and $\mu_i = 1$ for $i > l$. Then*

$$\partial_1^{N'_{\lambda,1}} s_\mu([x_1] - [x_2]) = c_\lambda s_{(\mu_1, 1^{l'-1})}([x_1] - [x_2]).$$

(iv) *For $m, n \geq 1$ we have*

$$a_{(m, 1^{n-1})}([x_1] - [x_2]) = (-1)^{n-1} x_1^{m-1} x_2^{n-1} (x_1 - x_2).$$

Proof. Notice that

$$\partial_1 s_\lambda(t) = \sum_{i=1}^l [w_l, \dots, w_i - 1, \dots, w_1].$$

In the right hand side $[w_l, \dots, w_i - 1, \dots, w_1] \neq 0$ if and only if all its components are different. In terms of the diagram of λ , $\partial_1 s_\lambda(t)$ is a sum of $s_\mu(t)$ with μ being the diagram obtained from λ by removing one box. For example

$$\partial_1 s_{(2,2,1)}(t) = s_{(2,1,1)}(t) + s_{(2,2)}(t).$$

(i) Notice that $N'_{\lambda,1}$ is a number of boxes on two to l -th rows of the diagram of λ which are on the right of the first column. Thus if $n < N'_{\lambda,1}$ it is impossible to get the hook diagram by removing n boxes from λ . Then the assertion of (i) follows from Corollary 1.

(ii) There is only one hook diagram in diagrams obtained from λ by removing $N'_{\lambda,1}$ boxes. It is $\mu := (\lambda_1, 1^{l-1})$. Let us compute the coefficient c of $s_\mu(t)$ in $\partial_1^{N'_{\lambda,1}} s_\lambda(t)$. Consider Equation (21) with $N_{\lambda,k}$ being replaced by $N'_{\lambda,1}$. In the right hand side, $s_\mu(t)$ appears only as a term such that $r_l = 0$ and $(w_{l-1} - r_{l-1}, \dots, w_1 - r_1)$ is a permutation of $(l-1, \dots, 2, 1)$. Let us write, for $1 \leq i \leq l-1$,

$$w_i - r_i = \sigma(i), \quad \sigma \in S_{l-1}.$$

Then by a similar calculation to that in the proof of Theorem 2 we have

$$\frac{c}{N'_{\lambda,1}!} = \sum_{\sigma \in S_{l-1}} \frac{\text{sgn } \sigma}{(w_1 - \sigma(1))! \cdots (w_{l-1} - \sigma(l-1))!} = \frac{\prod_{i=1}^{l-1} (w_i - 1)!}{\prod_{i=1}^{l-1} (w_i - 1)!}.$$

(iii) Similarly to the proof of (ii) the only Schur function appearing in $\partial_1^{N'_{\lambda,1}} s_\mu(t)$ which does not vanish at $t = [x_1] - [x_2]$ is $s_\nu(t)$, $\nu = (\mu_1, 1^{l'-1})$. Let us compute the coefficient c' of $s_\nu(t)$ in $\partial_1^{N'_{\lambda,1}} s_\mu(t)$.

Let $(w'_{l'}, \dots, w'_1)$ be the strictly decreasing sequence corresponding to μ . Then

$$\partial_1^{N'_{\lambda,1}} s_\mu(t) = \sum_{r_1 + \dots + r_{l'} = N'_{\lambda,1}} \frac{N'_{\lambda,1}!}{r_1! \cdots r_{l'}!} [w'_{l'} - r_l, \dots, w'_1 - r_1]. \quad (35)$$

In the right hand side $[w'_{l'} - r_l, \dots, w'_1 - r_1]$ is proportional to $s_\nu(t)$ if and only if $r_i = 0$ for $i = l'$ or $i < l' - l$, and $(w'_{l'-1} - r_{l'-1}, \dots, w'_{l'-l+1} - r_{l'-l+1})$ is a permutation of $(l' - 1, l' - 2, \dots, l' - l + 1)$. Let us write

$$w'_i - r_i = \sigma(i), \quad l' - l + 1 \leq i \leq l' - 1, \quad \sigma \in S_{l-1}.$$

Then

$$\begin{aligned} \frac{c'}{N'_{\lambda,1}!} &= \sum_{\sigma \in S_{l-1}} \frac{\text{sgn } \sigma}{(w'_{l'-l+1} - \sigma(l' - l + 1))! \cdots (w'_{l'-1} - \sigma(l' - 1))!} \\ &= \frac{\prod_{l'-l+1 \leq i < j \leq l'-1} (w'_j - w'_i)}{\prod_{i=l'-l+1}^{l'-1} (w'_i - l' + l - 1)!}. \end{aligned} \quad (36)$$

Let us rewrite c' in terms of λ_j . By assumption $\mu_i = \lambda_i$ for $2 \leq i \leq l$ which implies

$$w'_i = \mu_{l'+1-i} + i - 1 = \lambda_{l'+1-i} + i - 1, \quad l' - l + 1 \leq i \leq l' - 1.$$

Substitute it into (36) and get

$$c' = \frac{N'_{\lambda,1}!}{\prod_{i=2}^l (\lambda_i + l - 1 - i)!} \prod_{2 \leq i < j \leq l} (\lambda_i - \lambda_j + j - i),$$

which equals to c_λ .

(iv) Set $\lambda = (m, 1^{n-1})$ in (34). Then, using $s_{(r)}([x]) = x^r$, we get the assertion of (iv). ■

3 τ -function

In this section we lift the properties of Schur functions which have been proved in the previous section to τ -functions.

Let \leq be the partial order on the set of partitions defined as follows. For two partitions $\lambda = (\lambda_1, \dots, \lambda_l)$, $\mu = (\mu_1, \dots, \mu_{l'})$, $\lambda \leq \mu$ if and only if $\lambda_i \leq \mu_i$ for all i .

For a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ we consider a function $\tau(t)$ given as a series of the form

$$\tau(t) = s_\lambda(t) + \sum_{\lambda < \mu} \xi_\mu s_\mu(t), \quad (37)$$

where $\xi_u \in \mathbb{C}$.

Example. Let X be a compact Riemann surface of genus $g \geq 1$, p_∞ a point of X , $w_1 < \dots < w_g$ the gap sequence at p_∞ and z a local coordinate at p_∞ . Embed the affine ring of $X \setminus \{p_\infty\}$ into Sato's universal Grassmann manifold (UGM) as in the paper [12]. Then the tau function corresponding to this point of UGM has the expansion of the form (37).

Proposition 4 *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ a partition, $\tau(t)$ be a function of the form (37) and $0 \leq k \leq l - 1$. Then, if $\text{wt } \alpha < N_{\lambda,k}$*

$$\partial^\alpha \tau \left(\sum_{i=1}^k [x_i] \right) = 0.$$

Proof. For $\mu = (\mu_1, \dots, \mu_{l'})$ satisfying $\lambda \leq \mu$ we have

$$\text{wt } \alpha < \sum_{i=k+1}^l \lambda_i \leq \sum_{i=k+1}^{l'} \mu_i.$$

Thus

$$\partial^\alpha s_\mu \left(\sum_{i=1}^k [x_i] \right) = 0,$$

by Proposition 3. The assertion of the proposition follows from (37). ■

For a function $\tau(t)$ of the form (37) and $1 \leq k \leq l$ let $\tau^{(k)}(t)$ be the function defined by

$$\tau^{(k)}(t) = s_{(\lambda_1, \dots, \lambda_k)}(t) + \sum_{\mu} \xi_{\mu} s_{(\mu_1, \dots, \mu_k)}(t),$$

where the sum in the right hand side is over all partitions $\mu = (\mu_1, \dots, \mu_l)$ such that $\lambda < \mu$ and $\mu_i = \lambda_i$ for $k+1 \leq i \leq l$. In particular $\tau^{(l)}(\sum_{i=1}^k [x_i]) = \tau(\sum_{i=1}^k [x_i])$. We set $\tau^{(0)}(t) = 1$.

Theorem 5 *Let $\lambda = (\lambda_1, \dots, \lambda_g)$ be the partition determined from a gap sequence of genus g , $\tau(t)$ a function of the form (37).*

(i) *We have, for $0 \leq k \leq g$,*

$$\partial_{a_1^{(k)}} \cdots \partial_{a_{m_k}^{(k)}} \tau \left(\sum_{i=1}^k [x_i] \right) = c_k \tau^{(k)} \left(\sum_{i=1}^k [x_i] \right),$$

where c_k is the same as in Theorem 1.

(ii) *We have, for $k \geq 1$,*

$$\tau^{(k)} \left(\sum_{i=1}^k [x_i] \right) = \tau^{(k-1)} \left(\sum_{i=1}^{k-1} [x_i] \right) x_k^{\lambda_k} + O(x_k^{\lambda_k+1}).$$

Proof. Let $\mu = (\mu_1, \dots, \mu_l)$ be a partition of length l such that $\lambda \leq \mu$. Then $l \geq g$ and

$$\text{wt} \left(\partial_{a_1^{(k)}} \cdots \partial_{a_{m_k}^{(k)}} \right) = \sum_{i=1}^{m_k} a_i^{(k)} = \sum_{i=k+1}^g \lambda_i \leq \sum_{i=k+1}^l \mu_i. \quad (38)$$

If the inequality in the right hand side is not an equality,

$$\partial_{a_1^{(k)}} \cdots \partial_{a_{m_k}^{(k)}} s_\mu \left(\sum_{i=1}^k [x_i] \right) = 0, \quad (39)$$

by Proposition 3. Therefore, if the left hand side of (39) does not vanish, $l = g$ and

$$\sum_{i=k+1}^g \mu_i = \sum_{i=k+1}^g \lambda_i.$$

Since $\lambda_i \leq \mu_i$ for any i , it implies $\mu_i = \lambda_i$ for $k+1 \leq i \leq g$. For such μ we have, by Theorem 1,

$$\partial_{a_1^{(k)}} \cdots \partial_{a_{m_k}^{(k)}} s_\mu \left(\sum_{i=1}^k [x_i] \right) = c_k s_{(\mu_1, \dots, \mu_k)} \left(\sum_{i=1}^k [x_i] \right).$$

The assertion (i) follows from this.

(ii) The assertion easily follows from (i) of Proposition 1 and the definition of $\tau^{(k)}(t)$. ■
Combining (i) and (ii) of Theorem 5 we have

Corollary 2 *Under the same assumption as in Theorem 5 we have, for $1 \leq k \leq g$,*

$$\partial_{a_1^{(k)}} \cdots \partial_{a_{m_k}^{(k)}} \tau \left(\sum_{i=1}^k [x_i] \right) = \frac{c_k}{c_{k-1}} \partial_{a_1^{(k-1)}} \cdots \partial_{a_{m_{k-1}}^{(k-1)}} \tau \left(\sum_{i=1}^{k-1} [x_i] \right) x_k^{\lambda_k} + O(x_k^{\lambda_k+1}).$$

Corresponding Theorem 2 we have

Theorem 6 *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition of length l , $\tau(t)$ a function of the form (37), $0 \leq k \leq l$. Then*

$$\partial_1^{N_{\lambda,k}} \tau \left(\sum_{i=1}^k [x_i] \right) = c'_{\lambda,k} \tau^{(k)} \left(\sum_{i=1}^k [x_i] \right),$$

where $c'_{\lambda,k}$ is the same as in Theorem 2.

Proof. The theorem can be proved in a similar manner to Theorem 5 using Theorem 2. ■

Corollary 3 *Under the same assumption as in Theorem 6 we have, for $1 \leq k \leq l$,*

$$\partial_1^{N_{\lambda,k}} \tau \left(\sum_{i=1}^k [x_i] \right) = \frac{c'_{\lambda,k}}{c'_{\lambda,k-1}} \partial_1^{N_{\lambda,k-1}} \tau \left(\sum_{i=1}^{k-1} [x_i] \right) x_k^{\lambda_k} + O(x_k^{\lambda_k+1}).$$

In order to state the properties for $\tau(t)$ corresponding to Theorem 4 let us introduce one more function $\tau_2(t)$ associated with $\tau(t)$ by

$$\tau_2(t) = s_{(\lambda_1, 1^{l'-1})}(t) + \sum_{\mu} \xi_{\mu} s_{(\mu_1, 1^{l'-1})}(t), \quad (40)$$

where the sum in the right hand side is over all partitions $\mu = (\mu_1, \dots, \mu_{l'})$ of length $l' \geq l$ satisfying $\lambda < \mu$, $\mu_i = \lambda_i$ for $2 \leq i \leq l$ and $\mu_i = 1$ for $i > l$.

Theorem 7 Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition of length l and $\tau(t)$ a function of the form (37).

(i) If $n < N'_{\lambda,1}$

$$\partial_1^n \tau([x_1] - [x_2]) = 0.$$

(ii) We have

$$\partial_1^{N'_{\lambda,1}} \tau([x_1] - [x_2]) = c_\lambda \tau_2([x_1] - [x_2]),$$

where c_λ is the same as in Theorem 4.

(iii) We have

$$\tau_2([x_1] - [x_2]) = (-1)^{\lambda_1-1} x_1^{\lambda_1-1} x_2^{l-1} (x_1 - x_2) (1 + \dots),$$

where \dots part is a series in x_1, x_2 containing only terms proportional to $x_1^i x_2^j$ with $i + j > 0$.

(iv) We have the expansion

$$\tau_2([x_1] - [x_2]) = (-1)^{l-1} \tau^{(1)}([x_1]) x_2^{l-1} + O(x_2^l).$$

Proof. (i) By (i) of Theorem 4 we have $\partial_1^n s_\lambda([x_1] - [x_2]) = 0$.

Suppose that $\lambda < \mu$ and $\mu = (\mu_1, \dots, \mu_{l'})$ is of length l' . Then, $l' \geq l$ and

$$n < \sum_{i=2}^l \lambda_i - (l-1) \leq \sum_{i=2}^l \mu_i + \sum_{i=l+1}^{l'} (\mu_i - 1) - (l-1) = \sum_{i=2}^{l'} \mu_i - (l' - 1).$$

Thus $\partial_1^n s_\mu([x_1] - [x_2]) = 0$ by Theorem 4 (i) and the assertion (i) is proved.

(ii) By (ii) of Theorem 4 we have

$$\partial_1^{N'_{\lambda,1}} \tau([x_1] - [x_2]) = c_\lambda s_{(\lambda_1, 1^{l-1})}([x_1] - [x_2]) + \sum_{\mu} \xi_\mu \partial_1^{N'_{\lambda,1}} s_\mu([x_1] - [x_2]). \quad (41)$$

Let us compute the second term in the right hand side of (41).

Suppose that $\mu > \lambda$, $\mu = (\mu_1, \dots, \mu_{l'})$ is of length l' and $\partial_1^{N'_{\lambda,1}} s_\mu([x_1] - [x_2]) \neq 0$. In such a case, similarly to the proof of Theorem 4 (ii), it can be shown that μ should be of the form $\mu = (\mu_1, \lambda_2, \dots, \lambda_l, 1^{l'-l})$. Then

$$\partial_1^{N'_{\lambda,1}} s_\mu([x_1] - [x_2]) = c_\lambda s_{(\mu_1, 1^{l'-1})}([x_1] - [x_2]),$$

by (iii) of Theorem 4. Thus the right hand side of (41) becomes $c_\lambda \tau_2([x_1] - [x_2])$.

(iii) This is a direct consequence of Theorem 4 (iv).

(iv) Substituting $t = [x_1] - [x_2]$ in $\tau_2(t)$ we have, by (iv) of Theorem 4,

$$\begin{aligned}\tau_2([x_1] - [x_2]) &= (-1)^{l-1} x_1^{\lambda_1-1} x_2^{l-1} (x_1 - x_2) + \sum \xi_\mu (-1)^{l'-1} x_1^{\mu_1-1} x_2^{l'-1} (x_1 - x_2) \\ &= (-1)^{l-1} (x_1^{\lambda_1} + \sum_{\mu} \xi_\mu x_1^{\mu_1}) x_2^{l-1} + O(x_2^l),\end{aligned}\tag{42}$$

where the sum in μ in the right hand side is over all partitions μ of the form $\mu = (\mu_1, \lambda_2, \dots, \lambda_l)$ with $\mu_1 > \lambda_1$. Then the term in the bracket in the right hand side of (42) is $\tau^{(1)}([x_1])$. Thus (iv) is proved. ■

4 σ -function

In this section we deduce properties of sigma functions from those of tau functions established in the previous section. To this end we briefly recall the definitions and properties of sigma functions.

Let (n, s) be a pair of relatively prime integers satisfying $2 \leq n < s$ and X the compact Riemann surface corresponding to the algebraic curve defined by

$$f(x, y) = 0, \quad f(x, y) = y^n - x^s - \sum_{ni+sj < ns} \lambda_{ij} x^i y^j. \tag{43}$$

We assume that the affine curve (43) is nonsingular. Then the genus of X is $g = 1/2(n-1)(s-1)$. The Riemann surface X is called an (n, s) curve [2]. It has a point ∞ over the point $x = \infty$.

For a meromorphic function F on X we denote by $\text{ord}_\infty F$ the order of a pole at ∞ . The variables x and y can be considered as meromorphic functions on X which satisfy

$$\text{ord}_\infty x = n, \quad \text{ord}_\infty y = s.$$

Let φ_i , $i \geq 1$, be monomials of x and y satisfying the conditions

$$\begin{aligned}\{\varphi_i \mid i \geq 1\} &= \{x^i y^j \mid i \geq 0, n > j \geq 0\}, \\ \text{ord}_\infty \varphi_i &< \text{ord}_\infty \varphi_{i+1}, \quad i \geq 1.\end{aligned}\tag{44}$$

For example $\varphi_1 = 1$, $\varphi_2 = x$.

The gap sequence $w_1 < \dots < w_g$ at ∞ of X is defined by

$$\{w_i\} = \mathbb{Z}_{\geq 0} \setminus \{\text{ord}_\infty \varphi_i \mid i \geq 1\}.$$

It becomes a gap sequence of type (n, s) defined in of Example 1 in section 2.

A basis of holomorphic one forms on X is given by

$$du_{w_i} := -\frac{\varphi_{g+1-i} dx}{f_y}, \quad 1 \leq i \leq g. \tag{45}$$

Let z be the local coordinate at ∞ such that

$$x = \frac{1}{z^n}, \quad y = \frac{1}{z^s} (1 + O(z)). \quad (46)$$

Then we have

$$du_{w_i} = z^{w_i-1} (1 + O(z)) dz. \quad (47)$$

We fix an algebraic fundamental form $\widehat{\omega}(p_1, p_2)$ on X [11] and decompose it as

$$\widehat{\omega}(p_1, p_2) = d_{p_2} \Omega(p_1, p_2) + \sum_{i=1}^g du_{w_i}(p_1) dr_i(p_2),$$

where

$$\Omega(p_1, p_2) = \frac{\sum_{i=0}^{n-1} y_1^i \left[\frac{f(z, w)}{w^{i+1}} \right]_{+(z, w)=(x_2, y_2)}}{(x_1 - x_2) f_y(x_1, y_1)} dx_1,$$

$$\left[\sum_{m=-\infty}^{\infty} a_m w^m \right]_+ = \sum_{m=0}^{\infty} a_m w^m.$$

Then dr_i automatically becomes a differential of the second kind whose only singularity is ∞ and $\{du_{w_i}, dr_i\}$ is a symplectic basis of $H^1(X, \mathbb{C})$ [11].

We take a symplectic basis of the homology group $H_1(X, \mathbb{Z})$ and define period matrices ω_i, η_i , $i = 1, 2$ by

$$2\omega_1 = \left(\int_{\alpha_j} du_{w_i} \right), \quad 2\omega_2 = \left(\int_{\beta_j} du_{w_i} \right),$$

$$-2\eta_1 = \left(\int_{\alpha_j} dr_i \right), \quad -2\eta_2 = \left(\int_{\beta_j} dr_i \right).$$

The normalized period matrix τ is given by $\tau = \omega_1^{-1} \omega_2$.

Let $\tau {}^t \delta' + {}^t \delta''$, $\delta', \delta'' \in \mathbb{R}^g$ be the Riemann's constant with respect to the choice $(\{\alpha_i, \beta_i\}, \infty)$, $\delta = {}^t(\delta', \delta'')$ and $\theta[\delta](z, \tau)$ the Riemann's theta function with the characteristic δ . The sigma function for these data is defined in [1] (see also [11]).

Definition 3 The sigma function $\sigma(u)$, $u = {}^t(u_{w_1}, \dots, u_{w_g})$ of an (n, s) curve X is defined by

$$\sigma(u) = C \exp \left(\frac{1}{2} {}^t u \eta_1 \omega_1^{-1} u \right) \theta[\delta]((2\omega_1)^{-1} u, \tau),$$

where C is a certain constant.

Let $\lambda = (\lambda_1, \dots, \lambda_g)$ be the partition corresponding to the gap sequence at ∞ of X . Then the constant C is specified by the condition that the expansion of $\sigma(u)$ at the origin is of the form

$$\sigma(u) = s_\lambda(t)|_{t_{w_i}=u_{w_i}} + \dots,$$

where \dots part is a series in u_{w_i} only containing terms proportional to $\prod u_{w_i}^{\alpha_i}$ with $\sum \alpha_i w_i > |\lambda|$.

For $m_i \in \mathbb{Z}^g$, $i = 1, 2$, the sigma function obeys the following transformation rule:

$$\begin{aligned} & \sigma\left(u + 2 \sum_{i=1}^2 \omega_i m_i\right) \\ &= (-1)^{t_{m_1 m_2 + 2(\delta' m_1 - \delta'' m_2)}} \exp\left(2 \sum_{i=1}^2 {}^t(\eta_i m_i) \left(u + \sum_{i=1}^2 \omega_i m_i\right)\right) \sigma(u). \end{aligned} \quad (48)$$

Let A be the affine ring of $X \setminus \{\infty\}$. As a vector space $\{\varphi_i | i \geq 1\}$ is a basis of A . We embed A into UGM using the local coordinate z as in [12]. Then the tau function $\tau(t)$ of the KP-hierarchy corresponding to this point of UGM has the form

$$\tau(t) = s_\lambda(t) + \sum_{\lambda < \mu} \xi_\mu s_\mu(t).$$

It can be expressed in terms of the sigma function as

$$\tau(t) = \exp\left(-\sum_{i=1}^{\infty} c_i t_i + \frac{1}{2} \widehat{q}(t)\right) \sigma(Bt), \quad (49)$$

where $\widehat{q}(t) = \sum \widehat{q}_{ij} t_i t_j$, $B = (b_{ij})_{1 \leq i \leq g, 1 \leq j}$ a certain $g \times \infty$ matrix satisfying the condition

$$b_{ij} = \begin{cases} 0 & \text{if } j < w_i \\ 1 & \text{if } j = w_i, \end{cases} \quad (50)$$

and $c_i, \widehat{q}_{ij}, b_{ij}$ are certain constants [12]¹ [4][5]. The constant c_i are irrelevant to c_k in Theorem 1 and is not used in other parts of this paper.

In this section ∂_i is used for $\partial/\partial t_i$ as in the previous section and ∂_{u_i} is used for $\partial/\partial u_i$.

A point $p \in X$ is identified with its Abel-Jacobi image $\int_{\infty}^p d\mathbf{u}$, where $d\mathbf{u} = {}^t(du_{w_1}, \dots, du_{w_g})$. By the definition of the matrix B , for $p \in X$, the following equation is valid:

$$B[z(p)] = p, \quad (51)$$

where $z(p)$ is the value of the local coordinate z at p and $[z(p)] = [z(p), z(p)^2/2, \dots]$ as before.

Corresponding to Proposition 4 we have

Theorem 8 *Let $0 \leq k \leq g-1$. If $\sum_{i=1}^g \alpha_i w_i < N_{\lambda, k}$ then*

$$\partial_{u_{w_1}}^{\alpha_1} \cdots \partial_{u_{w_g}}^{\alpha_g} \sigma\left(\sum_{i=1}^k p_i\right) = 0,$$

for $p_1, \dots, p_k \in X$.

¹In the defining equation of c_i in [12] c_i should be corrected to c_i/i

Remark 2 In the case of the curve $y^n = f(x)$ Theorem 8 is proved in [9].

Lemma 4 Let $0 \leq k \leq g - 1$. If $\text{wt } \alpha < N_{\lambda, k}$

$$\partial^\alpha \sigma(Bt)|_{t=t^{(k)}} = 0,$$

where $t^{(k)} = \sum_{i=1}^k [z_i]$, $z_i = z(p_i)$ and $p_1, \dots, p_k \in X$.

Proof. The assertion easily follows from (49) and Proposition 4. ■

Proof of Theorem 8.

We introduce the lexicographical order on $\mathbb{Z}_{\geq 0}^g$ comparing from the right. Namely define $(\alpha_1, \dots, \alpha_g) < (\beta_1, \dots, \beta_g)$ if there exists $1 \leq i \leq g$ such that $\alpha_g = \beta_g, \dots, \alpha_{i+1} = \beta_{i+1}$ and $\alpha_i < \beta_i$.

We prove

$$\partial_{u_{w_1}}^{\beta_1} \dots \partial_{u_{w_g}}^{\beta_g} \sigma(Bt^{(k)}) = 0, \quad \sum_{i=1}^g \beta_i w_i < N_{\lambda, k}, \quad (52)$$

by induction on the order of $(\beta_1, \dots, \beta_g)$.

The case $(\beta_1, \dots, \beta_g) = (0, \dots, 0)$ is obvious by Lemma 4.

Take $(\beta_1, \dots, \beta_g) > (0, \dots, 0)$ such that $\sum_{i=1}^g \beta_i w_i < N_{\lambda, k}$. Suppose that (52) is valid for any $(\beta'_1, \dots, \beta'_g)$ satisfying $(\beta'_1, \dots, \beta'_g) < (\beta_1, \dots, \beta_g)$ and $\sum_{i=1}^g \beta'_i w_i < N_{\lambda, k}$.

Notice that $\sigma(Bt)$ is a composition of $\sigma(u)$ with

$$u_{w_i} = t_{w_i} + \sum_{w_i < j} b_{ij} t_j, \quad 1 \leq i \leq g. \quad (53)$$

By the chain rule,

$$\partial_{w_i} = \partial_{u_{w_i}} + \sum_{l < i} b_{lw_i} \partial_{u_{w_l}}. \quad (54)$$

Let j be the maximum number such that $\beta_j \neq 0$. Then

$$\begin{aligned} & \partial_{w_1}^{\beta_1} \dots \partial_{w_g}^{\beta_g} \sigma(Bt) \\ &= \partial_{u_1}^{\beta_1} (\partial_{u_{w_2}} + b_{1w_2} \partial_{u_1})^{\beta_2} \dots (\partial_{u_{w_j}} + \sum_{l < j} b_{lw_j} \partial_{u_{w_l}})^{\beta_j} \sigma(Bt) \\ &= \partial_{u_1}^{\beta_1} \partial_{u_{w_2}}^{\beta_2} \dots \partial_{u_{w_j}}^{\beta_j} \sigma(Bt) + \dots, \end{aligned} \quad (55)$$

where \dots part contains terms of the form

$$\partial_{u_1}^{\gamma_1} \dots \partial_{u_{w_g}}^{\gamma_g} \sigma(Bt), \quad \sum_{i=1}^g \gamma_i w_i < N_{\lambda, k}, \quad (\gamma_1, \dots, \gamma_g) < (\beta_1, \dots, \beta_g).$$

At $t = t^{(k)}$ the left hand side of (55) vanishes by Lemma 4 and \dots part in the right hand side of (55) vanishes by the assumption of induction. Thus (52) is proved. ■

Corresponding to Theorem 5 and Corollary 2 we have

Corollary 4 *Let $1 \leq k \leq g$ and $p_1, \dots, p_k \in X$. Then*

(i) *We have*

$$\partial_{u_{a_1}^{(k)}} \cdots \partial_{u_{a_{m_k}^{(k)}}} \sigma\left(\sum_{i=1}^k p_i\right) = c_k S_{(\lambda_1, \dots, \lambda_k)}(z_1, \dots, z_k) + \cdots,$$

where \cdots part is a series in z_i containing only terms proportional to $\prod_{i=1}^k z_i^{\alpha_i}$ with $\sum_{i=1}^k \alpha_i > \sum_{i=1}^k \lambda_i$.

(ii) *The following expansion is valid:*

$$\partial_{u_{a_1}^{(k)}} \cdots \partial_{u_{a_{m_k}^{(k)}}} \sigma\left(\sum_{i=1}^k p_i\right) = \frac{c_k}{c_{k-1}} \partial_{u_{a_1}^{(k-1)}} \cdots \partial_{u_{a_{m_{k-1}}^{(k-1)}}} \sigma\left(\sum_{i=1}^{k-1} p_i\right) z_k^{\lambda_k} + O(z_k^{\lambda_k+1}).$$

Proof. By Theorem 8 and (54) we have

$$\partial_{a_1}^{(k)} \cdots \partial_{a_{m_k}}^{(k)} \sigma(Bt)|_{t=t^{(k)}} = \partial_{u_{a_1}^{(k)}} \cdots \partial_{u_{a_{m_k}}^{(k)}} \sigma\left(\sum_{i=1}^k p_i\right). \quad (56)$$

Let us write (49) as $\sigma(Bt) = \varepsilon(t)\tau(t)$ with

$$\varepsilon(t) = \exp\left(\sum_{i=1}^{\infty} c_i t_i - \frac{1}{2} \widehat{q}(t)\right).$$

By Proposition 4 and Corollary 2 we have

$$\begin{aligned} & \partial_{a_1}^{(k)} \cdots \partial_{a_{m_k}}^{(k)} \sigma(Bt)|_{t=t^{(k)}} \\ &= \varepsilon(t^{(k)}) \partial_{a_1}^{(k)} \cdots \partial_{a_{m_k}}^{(k)} \tau(t^{(k)}) \\ &= c_{k-1}^{-1} c_k \varepsilon(t^{(k)}) \partial_{a_1}^{(k-1)} \cdots \partial_{a_{m_{k-1}}}^{(k-1)} \tau(t^{(k-1)}) z_k^{\lambda_k} + O(z_k^{\lambda_k+1}) \\ &= c_{k-1}^{-1} c_k \partial_{a_1}^{(k-1)} \cdots \partial_{a_{m_{k-1}}}^{(k-1)} \sigma(Bt)|_{t=t^{(k-1)}} z_k^{\lambda_k} + O(z_k^{\lambda_k+1}). \end{aligned} \quad (57)$$

Then the assertion (ii) follows from (56) and the assertion (i) follows from the second line of (57), Theorem 5 (i) and the definition of $\tau^{(k)}(t)$. ■

The following corollary can similarly be proved using Theorem 6 and Corollary 3.

Corollary 5 *Let $1 \leq k \leq g$ and $p_1, \dots, p_k \in X$. Then*

(i) *We have*

$$\partial_{u_1}^{N_{\lambda,k}} \sigma\left(\sum_{i=1}^k p_i\right) = c'_{\lambda,k} S_{(\lambda_1, \dots, \lambda_k)}(z_1, \dots, z_k) + \cdots,$$

where \dots part is a series in z_i containing only terms proportional to $\prod_{i=1}^k z_i^{\alpha_i}$ with $\sum_{i=1}^k \alpha_i > \sum_{i=1}^k \lambda_i$.

(ii) The following expansion holds:

$$\partial_{u_1}^{N_{\lambda,k}} \sigma\left(\sum_{i=1}^k p_i\right) = \frac{c'_{\lambda,k}}{c'_{\lambda,k-1}} \partial_{u_1}^{N_{\lambda,k-1}} \sigma\left(\sum_{i=1}^{k-1} p_i\right) z_k^{\lambda_k} + O(z_k^{\lambda_k+1}).$$

Corresponding to Theorem 7 we have

Theorem 9 (i) If $n < N'_{\lambda,1}$ we have, for $p_1, p_2 \in X$,

$$\partial_{u_1}^n \sigma(p_1 - p_2) = 0.$$

(ii) The following expansion with respect to $z_i = z(p_i)$, $i = 1, 2$ is valid:

$$\partial_{u_1}^{N'_{\lambda,1}} \sigma(p_1 - p_2) = (-1)^{g-1} c_{\lambda}(z_1 z_2)^{g-1} (z_1 - z_2) (1 + \dots),$$

where \dots part is a series in z_1, z_2 which contains only terms proportional to $z_1^i z_2^j$ with $i + j > 0$.

(iii) We have

$$\partial_{u_1}^{N'_{\lambda,1}} \sigma(p_1 - p_2) = (-1)^{g-1} \frac{c_{\lambda}}{c'_{\lambda,1}} \partial_{u_1}^{N_{\lambda,1}} \sigma(p_1) z_2^{g-1} + O(z_2^g).$$

Proof. (i) Notice that

$$\partial_1^m \sigma(Bt) = \partial_{u_1}^m \sigma(Bt) \tag{58}$$

for any m . Differentiating $\sigma(Bt) = \varepsilon(t)\tau(t)$ and using (58) and (i) of Theorem 7 we have the assertion.

(ii) We have, by (i), (ii), (iii) of Theorem 7,

$$\begin{aligned} \partial_{u_1}^{N'_{\lambda,1}} \sigma(p_1 - p_2) &= \partial_1^{N'_{\lambda,1}} \sigma(Bt)|_{t=[z_1]-[z_2]} \\ &= \varepsilon([z_1] - [z_2]) \partial_1^{N'_{\lambda,1}} \tau([z_1] - [z_2]) \\ &= c_{\lambda} \varepsilon([z_1] - [z_2]) \tau_2([z_1] - [z_2]) \\ &= c_{\lambda} (-1)^{g-1} (z_1 z_2)^{g-1} (z_1 - z_2) (1 + \dots). \end{aligned}$$

(iii) In the computation in (ii) we have

$$\begin{aligned} c_{\lambda} \varepsilon([z_1] - [z_2]) \tau_2([z_1] - [z_2]) &= (-1)^{g-1} c_{\lambda} (c'_{\lambda,1})^{-1} \varepsilon([z_1] - [z_2]) \partial_1^{N_{\lambda,1}} \tau([z_1]) z_2^{g-1} + O(z_2^g) \\ &= (-1)^{g-1} c_{\lambda} (c'_{\lambda,1})^{-1} \partial_1^{N_{\lambda,1}} \sigma(Bt)|_{t=[z_1]} z_2^{g-1} + O(z_2^g), \end{aligned}$$

by Theorem 7 (i), (ii), (iv) and Theorem 6. Then the assertion follows from (58). ■

5 Addition formulae

Let $E(p_1, p_2)$ be the prime form [6, 10] of an (n, s) curve X . In [11] we have introduced the prime function $\tilde{E}(p_1, p_2)$ by

$$\tilde{E}(p_1, p_2) = -E(p_1, p_2) \prod_{i=1}^2 \sqrt{du_{w_g}(p_i)} \exp \left(\frac{1}{2} \int_{p_1}^{p_2} {}^t d\mathbf{u} \eta_1 \omega_1^{-1} \int_{p_1}^{p_2} d\mathbf{u} \right). \quad (59)$$

Notice that $\tilde{E}(p_1, p_2)$ is not a $(-1/2, -1/2)$ form but a multi-valued analytic function on X and thus it has a sense to talk about the transformation rule if p_i goes around a cycle of X .

The prime function has the following properties.

- (i) $\tilde{E}(p_2, p_1) = -\tilde{E}(p_1, p_2)$.
- (ii) As a function of p_1 , the zero divisor of $\tilde{E}(p_1, p_2)$ is $p_2 + (g-1)\infty$.
- (iii) Let $m_i = {}^t(m_{i1}, \dots, m_{ig}) \in \mathbb{Z}^g$. If p_2 goes round the cycle $\gamma = \sum_{i=1}^g (m_{1i}\alpha_i + m_{2i}\beta_i)$, $\tilde{E}(p_1, p_2)$ transforms as

$$\tilde{E}(p_1, \gamma p_2) = T(m_1, m_2 | \int_{p_1}^{p_2} d\mathbf{u}) \tilde{E}(p_1, p_2), \quad (60)$$

with

$$T(m_1, m_2 | u) = (-1)^{{}^t m_1 m_2 + 2(\delta' m_1 - \delta'' m_2)} \exp \left(2 \sum_{i=1}^2 {}^t (\eta_i m_i) (u + \sum_{i=1}^2 \omega_i m_i) \right).$$

- (iv) At (∞, ∞) , $\tilde{E}(p_1, p_2)$ has the expansion of the form

$$\tilde{E}(p_1, p_2) = (z_1 z_2)^{g-1} (z_1 - z_2) \left(1 + \sum_{i+j \geq 1} c_{ij} z_1^i z_2^j \right), \quad (61)$$

where $z_i = z(p_i)$.

The specialization $\tilde{E}(\infty, p)$ of $\tilde{E}(p_1, p_2)$ is defined by

$$-\tilde{E}(p_1, p_2) = \tilde{E}(\infty, p_2) z_1^{g-1} + O(z_1^g). \quad (62)$$

It has the following properties corresponding to (iii) and (iv) above.

- (iii)' Under the same notation as in (iii) for $\tilde{E}(p_1, p_2)$ we have

$$\tilde{E}(\infty, \gamma p_2) = T(m_1, m_2 | \int_{\infty}^{p_2} d\mathbf{u}) \tilde{E}(\infty, p_2). \quad (63)$$

- (iv)' $\tilde{E}(\infty, p_2) = z_2^g + O(z_2^{g+1})$.

The following theorem gives the expression of the prime function in terms of a derivative of the sigma function.

Theorem 10 *Let $\lambda = (\lambda_1, \dots, \lambda_g)$ be the partition corresponding to the gap sequence at ∞ of an (n, s) curve X . Then*

$$\tilde{E}(p_1, p_2) = (-1)^{g-1} c_{\lambda}^{-1} \partial_{u_1}^{N'_{\lambda,1}} \sigma(p_1 - p_2).$$

Lemma 5 *Under the same notation as in (60) we have*

$$\partial_{u_1}^{N'_{\lambda,1}} \sigma(p_1 - \gamma p_2) = T(m_1, m_2 | \int_{p_1}^{p_2} d\mathbf{u}) \partial_{u_1}^{N'_{\lambda,1}} \sigma(p_1 - p_2).$$

Proof. Notice that $\gamma p_2 = p_2 + 2\omega_1 m_1 + 2\omega_2 m_2$ and

$$\sigma(u - 2 \sum_{i=1}^2 \omega_i m_i) = T(-m_1, -m_2 | u) \sigma(u). \quad (64)$$

Applying $\partial_{u_1}^{N'_{\lambda,1}}$ to (64) and setting $u = p_1 - p_2$, we get, by Theorem 9 (i) we have

$$\partial_{u_1}^{N'_{\lambda,1}} \sigma(p_1 - p_2 - 2 \sum_{i=1}^2 \omega_i m_i) = T(-m_1, -m_2 | \int_{p_2}^{p_1} d\mathbf{u}) \partial_{u_1}^{N'_{\lambda,1}} \sigma(p_1 - p_2). \quad (65)$$

Then the assertion follows from $T(-m_1, -m_2 | u) = T(m_1, m_2 | -u)$. ■

Proof of Theorem 10.

Notice that

$$\partial_{u_1}^{N'_{\lambda,1}} \sigma(-u) = -\partial_{u_1}^{N'_{\lambda,1}} \sigma(u), \quad (66)$$

since $\sigma(-u) = (-1)^{|\lambda|} \sigma(u)$ [11] and $N'_{\lambda,1} = |\lambda| - 2g + 1$.

Consider the function

$$F(p_1, p_2) = \frac{\partial_{u_1}^{N'_{\lambda,1}} \sigma(p_1 - p_2)}{\tilde{E}(p_1, p_2)}. \quad (67)$$

It is symmetric in p_1 and p_2 by (66), (i) of properties of $\tilde{E}(p_1, p_2)$ and is a meromorphic function on $X \times X$ by Lemma 5. Fix p_1 near ∞ . As a function of p_2 $F(p_1, p_2)$ has no singularity by Theorem 8, Theorem 9 (ii) and the property (ii) of $\tilde{E}(p_1, p_2)$. Therefore it does not depend on p_2 . It means that, for some non-empty open neighborhood U of ∞ , $F(p_1, p_2)$ does not depend on p_2 on $U \times X$. Since $F(p_1, p_2)$ is symmetric, it is a constant on $U \times U$. Thus it is a constant on $X \times X$ because it is meromorphic. The constant can be determined by comparing the expansion using Theorem 9 (ii) and the property (iv) of $\tilde{E}(p_1, p_2)$. ■

Corollary 6 *For $p \in X$ we have*

$$\tilde{E}(\infty, p) = c'_{\lambda,1}{}^{-1} \partial_{u_1}^{N'_{\lambda,1}} \sigma(p).$$

Proof. Compare the expansion in the equation of Theorem 10 using (iii) of Theorem 9.

■

Remark 3 In the case of hyperelliptic curves the prime function can be given using the derivative determined from the sequence $a_j^{(2)}$. This is because $p_1 - p_2$ can be written as a sum $p_1 + p_2^*$ where $*$ denoting the hyperelliptic involution. Such expression is given in [7].

The following theorem had been proved in [11].

Theorem 11 [11] For $n \geq g$ and $p_i \in X$, $1 \leq i \leq n$,

$$\sigma\left(\sum_{i=1}^n p_i\right) = \frac{\prod_{i=1}^n \tilde{E}(\infty, p_i)^n}{\prod_{i < j} \tilde{E}(p_i, p_j)} \det(\varphi_i(p_j))_{1 \leq i, j \leq n}.$$

By comparing the top term of the series expansion in $z(p_n)$, using Theorem 2 and Corollary 5, beginning from $n = g$ successively in the equation of this theorem we get

Corollary 7 For $n < g$ we have

$$\partial_{u_1}^{N_{\lambda, n}} \sigma\left(\sum_{i=1}^n p_i\right) = c'_{\lambda, n} \frac{\prod_{i=1}^n \tilde{E}(\infty, p_i)^n}{\prod_{i < j} \tilde{E}(p_i, p_j)} \det(\varphi_i(p_j))_{1 \leq i, j \leq n}.$$

Combining Theorem 10 and Corollaries 6 and 7 we have the following addition formulae for sigma functions.

Corollary 8 (i) For $n \geq g$ and $p_i \in X$, $1 \leq i \leq n$,

$$\frac{\sigma\left(\sum_{i=1}^n p_i\right) \prod_{i < j} \partial_{u_1}^{N'_{\lambda, 1}} \sigma(p_j - p_i)}{\prod_{i=1}^n (\partial_{u_1}^{N_{\lambda, 1}} \sigma(p_i))^n} = b_{\lambda, n} \det(\varphi_i(p_j))_{1 \leq i, j \leq n},$$

with

$$b_{\lambda, n} = (-1)^{\frac{1}{2}gn(n-1)} c_{\lambda}^{\frac{1}{2}n(n-1)} (c'_{\lambda, 1})^{-n^2}.$$

(ii) For $n < g$

$$\frac{\partial_{u_1}^{N_{\lambda, n}} \sigma\left(\sum_{i=1}^n p_i\right) \prod_{i < j} \partial_{u_1}^{N'_{\lambda, 1}} \sigma(p_j - p_i)}{\prod_{i=1}^n (\partial_{u_1}^{N_{\lambda, 1}} \sigma(p_i))^n} = b_{\lambda, n} \det(\varphi_i(p_j))_{1 \leq i, j \leq n},$$

with

$$b_{\lambda, n} = (-1)^{\frac{1}{2}gn(n-1)} c_{\lambda}^{\frac{1}{2}n(n-1)} (c'_{\lambda, 1})^{-n^2} c'_{\lambda, n}.$$

Similarly, using Theorem 11, Theorem 1 and Corollary 4, we have

Corollary 9 For $n < g$ and $p_i \in X$, $1 \leq i \leq n$, we have

$$\partial_{u_{a_1^{(n)}}} \cdots \partial_{u_{a_{mn}^{(n)}}} \sigma\left(\sum_{i=1}^n p_i\right) = c_n \frac{\prod_{i=1}^n \tilde{E}(\infty, p_i)^n}{\prod_{i < j}^n \tilde{E}(p_i, p_j)} \det(\varphi_i(p_j))_{1 \leq i, j \leq n}.$$

Corollary 10 For $n < g$ and $p_i \in X$, $1 \leq i \leq n$, we have

$$\frac{\partial_{u_{a_1^{(n)}}} \cdots \partial_{u_{a_{mn}^{(n)}}} \sigma\left(\sum_{i=1}^n p_i\right) \prod_{i < j} \partial_{u_1}^{N'_{\lambda,1}} \sigma(p_j - p_i)}{\prod_{i=1}^n (\partial_{u_1}^{N_{\lambda,1}} \sigma(p_i))^n} = b'_{\lambda,n} \det(\varphi_i(p_j))_{1 \leq i, j \leq n},$$

with

$$b'_{\lambda,n} = (-1)^{\frac{1}{2}gn(n-1)} c_{\lambda}^{\frac{1}{2}n(n-1)} (c'_{\lambda,1})^{-n^2} c_n.$$

In the case of hyperelliptic curves $\partial_{u_1}^{N'_{\lambda,1}} \sigma(p_j - p_i)$ can be replaced by a constant multiple of " $a_j^{(2)}$ -derivative" as remarked before (Remark after Corollary 6). Then Corollaries 8, 10 recovers the formulae in [13].

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